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LETTER TO THE EDITOR

Mean field renormalisation group for the spin-$\frac{1}{2}$ anisotropic Heisenberg model

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Abstract. The mean field renormalisation group approach is applied to the spin-$\frac{1}{2}$ anisotropic Heisenberg model. The critical temperature and the critical exponent are obtained as a function of the anisotropy for the two-dimensional model on square, triangular, and hexagonal lattices. The three-dimensional model on a simple cubic lattice is also analysed.

The mean field renormalisation group (MFRG) method has been recently proposed by Indekeu *et al* (1982) for computing critical properties of lattice spin systems. It is based on the comparison of the behaviour of two finite systems (clusters) of different sizes N, N'. For the two systems one computes the magnetisation per spin $m_N(K, b)$ and $m_{N'}(K', b')$, where b and b', assumed to be very small, are a symmetry breaking boundary condition (mean field) acting at the boundary of the N- and N'-spin clusters respectively ($N' < N$). By imposing a scaling relation of the form $m_{N'}(K', b') = \gamma m_N(K, b)$ between such approximate magnetisations and assuming a similar scaling relation between the parameters b and b' one gets

$$\partial m_{N'}(K', 0)/\partial b' = \partial m_N(K, 0)/\partial b, \tag{1}$$

which is independent of γ. From equation (1), which is interpreted as a renormalisation recursion relation between the coupling constants K and K' for the two rescaled systems, critical fixed points K^* are extracted. The critical exponent of the correlation length, y, can also be obtained by computing

$$\mathrm{d}K'/\mathrm{d}K|_{K^*} = l^y \tag{2}$$

where $l = (N/N')^{1/d}$ is the rescaling factor and d is the dimensionality of the system.

The MFRG approach summarised above has been applied to a number of systems: classical and quantum pure spin systems (Indekeu *et al* 1982); classical random systems including the spin glass (Droz *et al* 1982); geometric phase transitions (De'Bell 1983); the triangular Ising antiferromagnet (Slotte 1984) and the disordered transverse Ising model (Plascak 1984). In several cases, quite good results are obtained using just the simplest choice for the clusters namely, one- and two-spin clusters respectively.

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In this letter we study the spin-$\frac{1}{2}$ anisotropic Heisenberg model described by the Hamiltonian

$$H = -J \sum_{(i,j)} [\sigma_i^z \sigma_j^z + \eta(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)], \tag{3}$$

where η ranges from 0 (Ising limit) to 1 (isotropic Heisenberg limit), J is the nearest neighbour exchange interaction, $\boldsymbol{\sigma}_i$ are Pauli spin matrices, and the sum runs over sites on a d-dimensional lattice. This model has been recently treated by using a real space renormalisation group procedure (Suzuki and Takano 1979, Stinchcombe 1981, Caride *et al* 1983). It has been shown that the critical temperature T_c of the ferromagnetic transition decreases as a function of the anisotropy. In particular, in the isotropic Heisenberg limit T_c vanishes for the two-dimensional model while, for the three-dimensional model T_c remains finite. Moreover, the vanishing of T_c for the two-dimensional model has been proved to be an exact result (Mermin and Wagner 1966).

In order to apply the MFRG approach to the model (3) we consider herein clusters up to the size where an analytical treatment can be done without great difficulty. The two-dimensional model on square, triangular, and hexagonal lattices, as well as the three-dimensional model on a simple cubic lattice, can then be treated by using the simple clusters shown in figure 1.

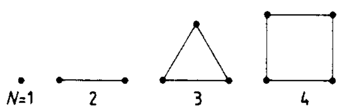

Figure 1. Clusters used in the present calculation. The spins are represented by the circles.

It is clear that, from equation (1) alone, one cannot determine the complete renormalisation flow diagram in the parameter space of the Hamiltonian (3) namely, $K = J/k_B T$ and η. One can, however, from equations (1) and (2), obtain the critical temperature, as well as estimates of the critical exponent y as a function of the anisotropy. As an example, we give below the equivalent of equations (1) and (2) in the case where $N' = 1$ and $N = 2$:

$$cK' = 2(c-1)K/(1 + e^{-2K} \cosh 2K\eta), \tag{4}$$

$$l^y = 1 + [cK\, e^{-2K}(\cosh 2K\eta - \eta \sinh 2K\eta)/(c-1)]_{K^*}, \tag{5}$$

where K^* is the fixed point solution of equation (4) for a given value of η and c is the coordination number of the particular lattice considered. Similar expressions can be obtained by using the bigger clusters of figure 1.

Figure 2 shows the critical temperature and the critical exponent as a function of the anisotropy for the two-dimensional model on a square lattice. It can be seen that an improvement is achieved by increasing the size of the clusters. In the Ising limit, the best values for the critical temperature and critical exponent (with $N' = 2$ and $N = 4$) are $k_B T_c/J = 2.70$ and $y = 0.78$ respectively, which should be compared with the exact values $k_B T_c/J = 2.27$ and $y = 1.0$. In the isotropic Heisenberg limit one has, in both cases, $T_c = 0$ and $y = 0$ which are the exact results (Mermin and Wagner 1966, Polyakov 1975). The continuous variation of y with the anisotropy is a consequence

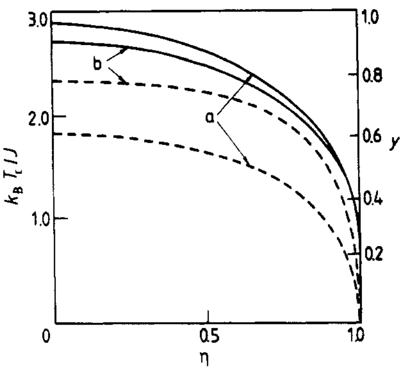

Figure 2. Critical temperature k_BT_c/J (full curve) and critical exponent y (broken curve) as a function of the anisotropy for the two-dimensional model on a square lattice. a, $N'=1$ and $N=2$. b, $N'=2$ and $N=4$.

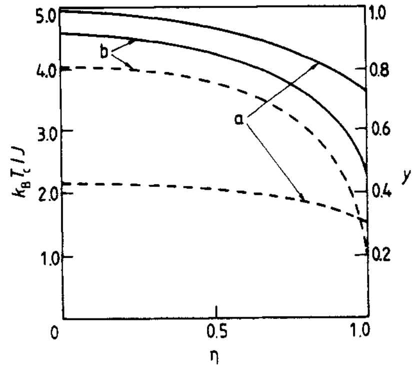

Figure 3. Critical temperature k_BT_c/J (full curve) and critical exponent y (broken curve) as a function of the anisotropy for the two-dimensional model on a triangular lattice. a, $N'=1$ and $N=2$. b, $N'=2$ and $N=3$.

of having only one recursion relation for the parameters of the system. However, the crossover from the Ising model to the isotropic Heisenberg model is apparent in this figure. One can then note that the present approach is capable of giving, in a single scheme, quite good results for the critical properties of the spin-$\frac{1}{2}$ anisotropic Heisenberg model on a square lattice.

Other types of lattice can also be treated within the present formalism by just taking into account properly the effects of the boundary conditions on each cluster. Figure 3 shows the critical temperature and the critical exponent as a function of the anisotropy for the two-dimensional model on a triangular lattice. An improvement is again achieved by considering clusters of bigger sizes. The best values in the Ising limit are $k_BT_c/J=4.56$ and $y=0.81$ which should be compared with the exact results $k_BT_c/J=3.64$ and $y=1.0$. In the isotropic Heisenberg limit the critical temperature and the critical exponent remain finite. However, by increasing the size of the clusters one can note that: (i) the cross-over between the Ising and isotropic Heisenberg models is clearly more pronounced and (ii) T_c and y tend to their exact values (zero).

The results for the two-dimensional model on an hexagonal lattice are shown in figure 4 in the case where $N'=1$ and $N=2$. In the Ising limit one has $k_BT_c/J=1.82$ and $y=0.70$. Although the critical temperature goes to zero as $\eta\rightarrow 1$, as expected, some spurious results are obtained in the vicinity of the isotropic Heisenberg limit: for $\eta\lesssim 1$, $dT_c/d\eta$ is finite and the critical exponent reaches a constant value $y=0.28$. Moreover, at $\eta=1$ equation (4) leads to

$$\tau'=\tfrac{9}{8}\tau, \qquad (6)$$

where $\tau'=K'^{-1}$ and $\tau=K^{-1}$. From (6) one has $y_\tau=0.34$. In this case, as well as for the model on a triangular lattice, the clusters are still too simple to give a good description of the model in the isotropic Heisenberg limit.

Finally, the present approach can easily be extended to the three-dimensional model. The results are shown in figure 5. The best values of the critical temperature and critical exponent in the Ising limit are $k_BT_c/J=4.86$ and $y=0.79$ which should be compared with series expansions results $k_BT_c/J=4.67$ and $y=1.59$ (Domb 1974). In the isotropic Heisenberg limit one has $k_BT_c/J=3.64$ and $y=0.59$. The real space

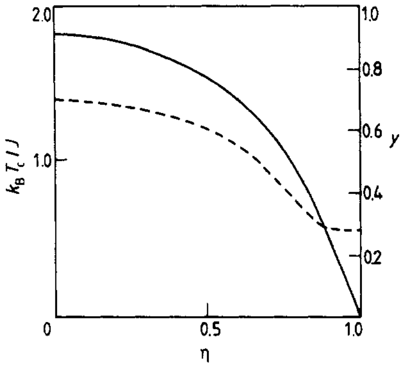

Figure 4. Critical temperature $K_B T_c/J$ (full curve) and critical exponent y (broken curve) as a function of the anisotropy for the two-dimensional model on a hexagonal lattice. In this case $N' = 1$ and $N = 2$.

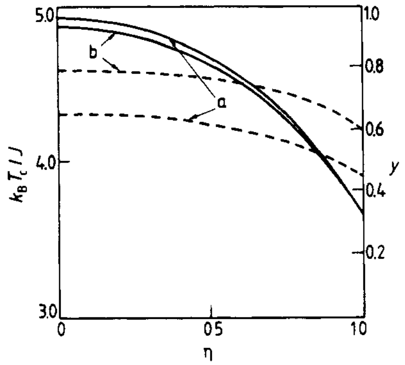

Figure 5. Critical temperature $K_B T_c/J$ (full curve) and critical exponent y (broken curve) as a function of the anisotropy for the three-dimensional model. a, $N' = 1$ and $N = 2$. b, $N' = 2$ and $N = 4$.

renormalisation group procedure by Stinchcombe (1981) gives, in this limit, $k_B T_c/J = 2.91$ and $y = 0.71$. One can note from figure 5 that only a slight improvement is obtained in this case by increasing the size of the clusters.

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